

# Random Matrix Theory with $U(N)$ Racah Algebra for Transition Strengths

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## Abstract.

For finite quantum many-particle systems, a given system, induced by a transition operator, makes transitions from its states to the states of the same system or to those of another system. Examples are electromagnetic transitions (then the initial and final systems are same), nuclear beta and double beta decay (then the initial and final systems are different), particle addition to or removal from a given system and so on. Working towards developing a complete statistical theory for transition strength densities (transition strengths multiplied by the density of states at the initial and final energies), we have started a program to derive formulas for the lower order bivariate moments of the strength densities generated by a variety of transition operators. In this paper results are presented for a transition operator that removes  $k_0$  number of particle by considering  $m$  spinless fermions in  $N$  single particle states. The Hamiltonian that is  $k$ -body is represented by EGUE( $k$ ) [embedded Gaussian unitary ensemble of  $k$ -body interactions] and similarly the transition operator by an appropriate independent EGUE. Employing the embedding  $U(N)$  algebra, finite- $N$  formulas for moments up to order four are derived and they show that in general the smoothed (with respect to energy) bivariate transition strength densities take bivariate Gaussian form. Extension of these results to particle addition operator and beta decay type operators are discussed.

## 1. Introduction

Let us begin with the statement from the preface to the proceedings of the meeting held in 2006 on "Applications of Random Matrices in Physics": [1] *Random matrices are widely and successfully used in physics for almost 60-70 years, beginning with the works of Wigner and Dyson. Initially proposed to describe statistics of levels in complex nuclei, the Random Matrix Theory has grown far beyond nuclear physics, and also far beyond just level statistics. It is constantly developing into new areas of physics and mathematics, and now constitutes a part of the general culture and curriculum of a theoretical physicist.* Besides applications in all branches of quantum physics, RMT is being used in disciplines such as Econophysics, Wireless communication, information theory, multivariate statistics, number theory, neural and biological networks and so on. The focus in this article is on the frontier topic of statistical properties of isolated finite many-particle quantum systems [2]. Examples for these systems are atoms, atomic nuclei, mesoscopic systems (quantum dots, small metallic grains), interacting spin systems modeling quantum computing core, ultra-cold atoms and so on. A route to investigate statistical properties is to employ the classical GOE or GUE or GSE random matrix ensembles with various deformations. For these, as Wigner states: *the assumption is that the Hamiltonian*

which governs the behavior of a complicated system is a random real symmetric or complex Hermitian or Quaternion real matrix, with no special properties except for its symmetric or Hermitian or Quaternion real nature. However, for most of the isolated finite many-particle quantum systems, their constituents predominantly interact via two-particle interactions and the classical random matrix ensembles are too unspecific to account for this most important feature. One refinement which retains the basic stochastic approach but allows for this feature consists in the use of embedded random matrix ensembles [3, 4, 5, 6, 7, 8].

Therefore, it is more appropriate to represent an isolated finite interacting quantum system, say with  $m$  particles (fermions or bosons) in  $N$  single particle (sp) states by random matrix models generated by random  $k$ -body (note that  $k < m$  and most often we have  $k = 2$ ) interactions and propagate the information in the interaction to many particle spaces. Thus we have random interaction matrix models for  $m$ -particle systems. In the simplest version, the  $k$ -particle Hamiltonian ( $H$ ) of a spinless fermion (or boson) system is represented by GOE/GUE/GSE and then the  $m$  particle  $H$  matrix is generated using the  $m$ -particle Hilbert space geometry. The key element here is the recognition that the  $U(N)$  Lie algebra transports the information in the two-particle spaces to many-particle spaces. As a GOE/GUE/GSE random matrix ensemble in two-particle spaces is embedded in the  $m$ -particle  $H$  matrix, these ensembles are generically called embedded ensembles (EE). With GUE embedding we have EGUE and in this paper EGUE is used throughout. For EE, a general formulation for deriving analytical results is to use the Wigner-Racah algebra of the embedding Lie algebra [8]. The focus in the present paper is on transition strengths and they are not yet studied in any detail using EE [8]. Also, as emphasized in [4, 9], there are many open questions in the random matrix theory for transition strengths in finite interacting quantum many-particle systems.

For finite quantum many-particle systems, induced by a transition operator, a given system makes transitions from its states to the states of the same system or to the states of another system. Examples are electromagnetic transitions (then the initial and final systems are same), nuclear beta and double beta decay (then the initial and final systems are different), particle addition to or removal from a given system and so on. Given a state with energy  $E_i$  and say it is connected to a state  $E_f$  by a transition operator  $\mathcal{O}$ , then the transition strengths are  $|\langle E_f | \mathcal{O} | E_i \rangle|^2$  and these will determine for example the life time of a state. It is important to recognize that the transition strengths probe the structure of the eigenfunctions of a quantum many-body system and thus they are very important from the point of view of experiments probing the structure of a system. Also, they are needed in many applications (for example, beta decay transition strengths are essential for nucleosynthesis studies). In the statistical theories, it is more useful to deal with the corresponding transition strength density (this will take into account degeneracies in the eigenvalues) defined by

$$I_{\mathcal{O}}(E_i, E_f) = I(E_f) |\langle E_f | \mathcal{O} | E_i \rangle|^2 I(E_i) . \quad (1)$$

In Eq. (1),  $I(E)$  are state densities normalized to the dimension of the  $m$  particle spaces. Note that  $E_i$  and  $E_f$  belong to the same  $m$  particle system or different systems depending on the nature of  $\mathcal{O}$ , the transition operator.

Working towards developing a complete statistical theory for transition strength densities (transition strengths multiplied by the density of states at the initial and final energies) for isolated finite many-particle quantum systems, we have started a program to derive formulas for the lower order bivariate moments of the strength densities generated by a variety of transition operators. In general, the Hamiltonian may have many symmetries with the fermions (or bosons) carrying other degrees of freedom such as spin, orbital angular momentum, isospin and so on. Also we may have in the system different types of fermions (or bosons) and for example in atomic nuclei we have protons and neutrons. In addition, a transition operator may preserve particle number and other quantum numbers or it may change them. Among all these various situations,

in our program we have considered five different systems: (i) a system of  $m$  spinless fermions and a transition operator that preserves the particle number; (ii) a system of  $m$  spinless fermions and a transition operator that removes say  $k_0$  number of particles from the  $m$  fermion system; (iii) same as (ii) but for particles addition operator; (iv) a system with two types of spinless fermions with the transition operator changing  $k_0$  number of particles of one type to  $k_0$  number of other particles as in nuclear beta and double beta decay; (v) same as (i)-(iii) but for spinless boson systems. In [10], results are presented for (i). In the present paper results are presented for (ii) and their extensions to (iii) and (iv) are briefly discussed; Ref. [11] gives results from a first study for (iv). Let us add that (ii) and (iii) are important for example in nuclear physics as one and two particle removal and addition to a nucleus are important experimental probes of the structure of the atomic nucleus; see for example [12, 13]. Thus, the results in Sections 3-5 have applications in nuclear physics. Now we will give a preview.

Section 2 gives some basic results for EGUE( $k$ ) for spinless fermion systems as derived in [14]. Using these results, formulas for the lower order bivariate moments of the transition strength densities for the situation (ii) above are derived and they are presented in detail in Section 3. Using these, results in the asymptotic limit are derived and they are presented in Section 4. Extension of the results in Section 3 to the situations (iii) and (iv) above are briefly discussed in Section 5. Finally, in Section 6 gives conclusions and future outlook.

## 2. Basic EGUE( $k$ ) results for a spinless fermion system

Let us consider  $m$  spinless fermions in  $N$  degenerate sp states with the Hamiltonian  $\hat{H}$  a  $k$ -body operator,

$$\hat{H} = \sum_{i,j} V_{ij}(k) A_i^\dagger(k) A_j(k), \quad V_{ij}(k) = \langle k, i | \hat{H} | k, j \rangle. \quad (2)$$

Here  $A_i^\dagger(k)$  is a  $k$  particle (normalized) creation operator and  $A_i(k)$  is the corresponding annihilation operator (a hermitian conjugate). Also,  $i$  and  $j$  are  $k$ -particle indices. The  $k$  and  $m$  particle space dimensions are  $\binom{N}{k}$  and  $\binom{N}{m}$  respectively. Representing the  $V$  matrix, defined by the matrix elements  $V_{ij}$ , by GUE we have EGUE( $k$ ) for the  $H$  matrix in  $m$ -particle spaces. Note that, for  $V$  a GUE, the real and imaginary parts of  $V_{ij}$  are independent zero centered Gaussian random variables with variance satisfying,

$$\overline{V_{ab}(k) V_{cd}(k)} = V_H^2 \delta_{ad} \delta_{bc}. \quad (3)$$

Here the 'over-line' indicates ensemble average. From now on we will drop the hat over  $H$  and denote when needed  $H$  by  $H(k)$ . In physical systems in general  $k = 2$  but in some systems such as atomic nuclei and BEC it is possible to have  $k = 3$  and even  $k = 4$  [15, 16, 17].

The  $U(N)$  algebra that generates the embedding, as shown in [14], gives formulas for the lower order moments of the one-point function, the eigenvalue density  $I(E) = \overline{\langle \delta(H - E) \rangle}$  and also for the two-point function in the eigenvalues. Used here is the  $U(N)$  tensorial decomposition of the  $H(k)$  operator giving  $\nu = 0, 1, \dots, k$  irreducible parts  $B^{\nu, \omega_\nu}(k)$  and then,

$$H(k) = \sum_{\nu=0; \omega_\nu \in \nu}^k W_{\nu, \omega_\nu}(k) B^{\nu, \omega_\nu}(k). \quad (4)$$

Note that  $\omega_\nu$  are labels of the irreducible representations (irreps) of the subalgebras of  $U(N)$  and their explicit structure will not play any role in the present work. With the GUE( $k$ ) representation for the  $H(k)$  operator, the expansion coefficients  $W$ 's will be independent zero centered Gaussian random variables with

$$\overline{W_{\nu_1, \omega_{\nu_1}}(k) W_{\nu_2, \omega_{\nu_2}}(k)} = V_H^2 \delta_{\nu_1, \nu_2} \delta_{\omega_{\nu_1} \omega_{\nu_2}}. \quad (5)$$

For deriving formulas for the various moments, the first step is to apply the Wigner-Eckart theorem for the matrix elements of  $B^{\nu, \omega_\nu}(k)$ . Given the  $m$ -fermion states  $|f_m v_i\rangle$ , we have with respect to the  $U(N)$  algebra,  $f_m = \{1^m\}$ , the antisymmetric irrep in Young tableaux notation and  $v_i$  are additional labels. Note that the  $\nu$  label used for  $B$ 's corresponds to the Young tableaux  $\{2^\nu 1^{N-2\nu}\}$ . Now, Wigner-Eckart theorem for  $U(N) \supset G$  (here  $G$  is some subalgebra of  $U(N)$  giving  $v$  and  $\omega$  labels) gives

$$\langle f_m v_f | B^{\nu, \omega_\nu}(k) | f_m v_i \rangle = \langle f_m || B^\nu(k) || f_m \rangle C_{f_m v_f, \overline{f_m v_i}}^{\nu, \omega_\nu}. \quad (6)$$

Here,  $\langle - || - || - \rangle$  is the reduced matrix element and  $C_{---}^{---}$  is a  $U(N) \supset G$  Clebsch-Gordan (C-G) coefficient [note that we are not making a distinction between  $U(N)$  and  $SU(N)$ ]. Also, if  $|f_r v\rangle$  represents a state for  $r$  number of fermions, then  $|\overline{f_r v}\rangle$  represents the corresponding state for  $r$  number of holes or  $N - r$  number of fermions (see [14] for details). In Young tableaux notation  $\overline{f_m} = \{1^{N-m}\}$ . Definition of  $B^{\nu, \omega_\nu}(k)$  and the  $U(N)$  Wigner-Racah algebra will give,

$$\begin{aligned} |\langle f_m || B^\nu(k) || f_m \rangle|^2 &= \Lambda^\nu(N, m, m - k), \\ \Lambda^\mu(N', m', r) &= \binom{m' - \mu}{r} \binom{N' - m' + r - \mu}{r}. \end{aligned} \quad (7)$$

The  $\Lambda^\nu(N, m, k)$  are nothing but, apart from a  $N$  and  $m$  dependent factor, a  $U(N)$  Racah coefficient [14]. This and the various properties of the  $U(N)$  Wigner and Racah coefficients give two formulas for the ensemble average of a product any two  $m$  particle matrix elements of  $H$ ,

$$\begin{aligned} &\overline{\langle f_m v_1 | H(k) | f_m v_2 \rangle \langle f_m v_3 | H(k) | f_m v_4 \rangle} \\ &= V_H^2 \sum_{\nu=0; \omega_\nu}^k \Lambda^\nu(N, m, m - k) C_{f_m v_1, \overline{f_m v_2}}^{\nu, \omega_\nu} C_{f_m v_3, \overline{f_m v_4}}^{\nu, \omega_\nu}, \end{aligned} \quad (8)$$

and also

$$\begin{aligned} &\overline{\langle f_m v_1 | H(k) | f_m v_2 \rangle \langle f_m v_3 | H(k) | f_m v_4 \rangle} \\ &= V_H^2 \sum_{\nu=0; \omega_\nu}^{m-k} \Lambda^\nu(N, m, k) C_{f_m v_1, \overline{f_m v_4}}^{\nu, \omega_\nu} C_{f_m v_3, \overline{f_m v_2}}^{\nu, \omega_\nu} \end{aligned} \quad (9)$$

Eq. (9) follows by applying a Racah transform to the product of the two C-G coefficients appearing in Eq. (8). Let us mention two important properties of the  $U(N)$  C-G coefficients that are quite useful,

$$\sum_{v_i} C_{f_m v_i, \overline{f_m v_i}}^{\nu, \omega_\nu} = \sqrt{\binom{N}{m}} \delta_{\nu, 0}, \quad C_{f_m v_i, \overline{f_m v_j}}^{0, 0} = \binom{N}{m}^{-1/2} \delta_{v_i, v_j}. \quad (10)$$

From now on we will use the symbol  $f_m$  only in the C-G coefficients, Racah coefficients and the reduced matrix elements. However, for the matrix elements of an operator we will use  $m$  implying totally antisymmetric state for fermions. An important by-product of Eqs. (9) and (10) is

$$\sum_{v_j} \overline{\langle m v_i | H(k) | m v_j \rangle \langle m v_j | H(k) | m v_k \rangle} = \overline{[H(k)]^2}^m \delta_{v_i, v_k} \quad (11)$$

and we will use this in Section 3.

Starting with Eq. (4) and using Eqs. (5), (9) and (10) will immediately give the formula,

$$\overline{[H(k)]^2}^m = \binom{N}{m}^{-1} \sum_{v_i} \overline{\langle m v_i | [H(k)]^2 | m v_i \rangle} = V_H^2 \Lambda^0(N, m, k). \quad (12)$$

Similarly, for  $\overline{\langle H^4 \rangle^m}$  first the ensemble average is decomposed into 3 terms as,

$$\begin{aligned}
\overline{\langle [H(k)]^4 \rangle^m} &= \sum_{v_i} \overline{\langle mv_i | [H(k)]^4 | mv_i \rangle} \\
&= \sum_{v_i, v_j, v_p, v_l} \left[ \overline{\langle mv_i | H(k) | mv_j \rangle \langle mv_j | H(k) | mv_p \rangle \langle mv_p | H(k) | mv_l \rangle \langle mv_l | H(k) | mv_i \rangle} \right. \\
&\quad + \overline{\langle mv_i | H(k) | mv_j \rangle \langle mv_l | H(k) | mv_i \rangle \langle mv_j | H(k) | mv_p \rangle \langle mv_p | H(k) | mv_l \rangle} \\
&\quad \left. + \overline{\langle mv_i | H(k) | mv_j \rangle \langle mv_p | H(k) | mv_l \rangle \langle mv_j | H(k) | mv_p \rangle \langle mv_l | H(k) | mv_i \rangle} \right] . \tag{13}
\end{aligned}$$

Note that the trace  $\langle \langle H^4 \rangle \rangle^m = \binom{N}{m} \langle H^4 \rangle^m$ . It is easy to see that the first two terms simplify to give  $2[\langle H^2 \rangle^m]^2$  and the third term is simplified by applying Eq. (8) to the first ensemble average and Eq. (9) to the second ensemble average. Then, the final result is

$$\overline{\langle [H(k)]^4 \rangle^m} = 2 \left[ \overline{\langle H^2 \rangle^m} \right]^2 + V_H^4 \binom{N}{m}^{-1} \sum_{\nu=0}^{\min(k, m-k)} \Lambda^\nu(N, m, k) \Lambda^\nu(N, m, m-k) d(N : \nu) ; \tag{14}$$

$$d(N : \nu) = \binom{N}{\nu}^2 - \binom{N}{\nu-1}^2 . \tag{15}$$

Now, we will derive the formulas for the moments of the transition strength densities generated by a transition operator  $\mathcal{O}$  that removes  $k_0$  number of particles from a  $m$ -particle system.

### 3. Lower-order moments of transition strength densities: results for particle removal operators

Particle removal (or addition) operators are of great interest in nuclear physics. For example one particle (proton or neutron) removal from a target nucleus gives information about the single particle levels in the target and similarly, two-particle removal gives information about pairing force. Let us begin with a particle removal operator  $\mathcal{O}$  and say it removes  $k_0$  number of particles when acting on a  $m$  fermion state. Then the general form of  $\mathcal{O}$  is,

$$\mathcal{O} = \sum_{\alpha_0} V_{\alpha_0} A_{\alpha_0}(k_0) . \tag{16}$$

Here,  $A_{\alpha_0}(k_0)$  is a  $k_0$  particle annihilation operator and  $\alpha_0$  are indices for a  $k_0$  particle state. Note that  $A_{\alpha_0}(k_0)$  transforms as  $\{\overline{f_{k_0}}\} = \{1^{N-k_0}\}$  with respect to  $U(N)$  and  $A_{\alpha_0}^\dagger(k_0)$  transforms as  $\{f_{k_0}\}$ . It is important to recognize that the  $\mathcal{O}$  matrices will be rectangular matrices connecting  $m$  particle states to  $m - k_0$  particle states. In the defining space, the matrix will be a  $1 \times d_0$  matrix with matrix elements given by  $V_{\alpha_0}$ . Note that  $\alpha_0$  takes  $d_0$  values and  $d_0 = \binom{N}{k_0}$ . We will represent  $\mathcal{O}$  by EGUE implying that the defining space matrix elements  $V_{\alpha_0}$  are zero centered independent Gaussian random variables [also they are independent of the  $V_{ij}(k)$  variables in Eq. (2) and therefore also independent of the  $W$  variables in Eq. (4)] with variance satisfying

$$\overline{V_\alpha V_\beta^\dagger} = V_{\mathcal{O}}^2 \delta_{\alpha\beta} . \tag{17}$$

In many particle spaces the  $\mathcal{O}$  matrix will be a  $d_1 \times d_2$  matrix connecting  $d_1 = \binom{N}{m}$  number of  $m$ -particle states to  $d_2 = \binom{N}{m-k_0}$  number of  $(m - k_0)$ -particle states. Using Eqs. (16) and (17),

we have

$$\overline{\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m} = V_{\mathcal{O}}^2 \binom{m}{k_0}, \quad \overline{\langle \mathcal{O} \mathcal{O}^\dagger \rangle^m} = V_{\mathcal{O}}^2 \binom{N-m}{k_0}. \quad (18)$$

Similarly, Eq. (11) gives the relations,

$$\overline{\langle \mathcal{O}^\dagger \mathcal{O} H^p \rangle^m} = \overline{\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m} \overline{\langle H^p \rangle^m}, \quad \overline{\langle \mathcal{O}^\dagger H^p \mathcal{O} \rangle^m} = \overline{\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m} \overline{\langle H^p \rangle^{m-k_0}}. \quad (19)$$

Another useful result follows by introducing complete set of states between the  $\mathcal{O}^\dagger$  and  $\mathcal{O}$  operators in Eq. (18) and applying the Wigner-Eckart theorem,

$$\langle m \parallel A^\dagger(k_0) \parallel m - k_0 \rangle \langle m - k_0 \parallel A(k_0) \parallel m \rangle = \binom{N-k_0}{m-k_0}. \quad (20)$$

Following the procedure used in Section 2, it is possible to derive formulas for the lower order moments of the transition strength densities generated by  $\mathcal{O}$  defined by Eq. (16). The bivariate moments are defined by

$$M_{PQ} = \overline{\langle \mathcal{O}^\dagger H^Q \mathcal{O} H^P \rangle^m}. \quad (21)$$

and we will consider the moments  $P+Q=2$  and 4 (the  $P+Q=3$  moments are zero as we are using independent EGUE representations for  $\mathcal{O}$  and  $H$  matrices).

Firstly, Eqs. (19) gives,

$$\begin{aligned} M_{20} &= \overline{\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m} \overline{\langle H^2 \rangle^m}, & M_{02} &= \overline{\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m} \overline{\langle H^2 \rangle^{m-k_0}}, \\ M_{40} &= \overline{\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m} \overline{\langle H^4 \rangle^m}, & M_{04} &= \overline{\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m} \overline{\langle H^4 \rangle^{m-k_0}}. \end{aligned} \quad (22)$$

Now, Eq. (18) along with Eqs. (12) and (14) will give the formulas for  $M_{20}$ ,  $M_{02}$ ,  $M_{40}$  and  $M_{04}$ . Formula for the first non-trivial moment  $M_{11} = \overline{\langle \mathcal{O}^\dagger H \mathcal{O} H \rangle^m}$  is derived by introducing complete set of states between  $\mathcal{O}^\dagger$  and  $H$ ,  $H$  and  $\mathcal{O}$  and  $\mathcal{O}$  and  $H$  in the trace giving,

$$\begin{aligned} \binom{N}{m} M_{11}(m) &= \binom{N}{m} \overline{\langle \mathcal{O}^\dagger H \mathcal{O} H \rangle^m} = \sum_{v_1, v_2, v_3, v_4} \overline{\langle m, v_1 \mid \mathcal{O}^\dagger \mid m - k_0, v_2 \rangle \langle m - k_0, v_3 \mid \mathcal{O} \mid m, v_4 \rangle} \\ &\times \overline{\langle m - k_0, v_2 \mid H \mid m - k_0, v_3 \rangle \langle m, v_4 \mid H \mid m, v_1 \rangle}. \end{aligned} \quad (23)$$

Using Eq. (16) and applying Eq. (17) along with Eqs. (4) - (7) and the Wigner-Eckart theorem will give,

$$\begin{aligned} M_{11}(m) &= V_{\mathcal{O}}^2 V_H^2 \binom{N}{m}^{-1} \binom{N-k_0}{m-k_0} \sum_{\nu=0}^k [\Lambda^\nu(N, m - k_0, m - k_0 - k) \Lambda^\nu(N, m, m - k)]^{1/2} \\ &\times \sum_{v_1, v_2, v_3, v_4; \alpha; \omega_\nu} C_{f_m v_1, \overline{f_{m-k_0} v_2}}^{f_{k_0}, \alpha} C_{f_{m-k_0} v_3, \overline{f_m v_4}}^{f_{k_0}, \bar{\alpha}} C_{f_{m-k_0} v_2, \overline{f_m v_3}}^{\nu, \omega_\nu} C_{f_m v_4, \overline{f_m v_1}}^{\nu, \omega_\nu}. \end{aligned} \quad (24)$$

Simplifying the four C-G coefficients will give finally,

$$\begin{aligned} M_{11}(m) &= V_{\mathcal{O}}^2 V_H^2 \binom{N}{m}^{-1} \binom{N-k_0}{m-k_0} \sum_{\nu=0}^k Z_{11}(N, m, k_0, k, \nu); \\ Z_{11}(N, m, k_0, k, \nu) &= \left[ \binom{N}{k_0} d(N : \nu) \Lambda^\nu(N, m, m - k) \Lambda^\nu(N, m - k_0, m - k_0 - k) \right]^{1/2} \\ &\times (-1)^{\phi(f_m, \overline{f_{m-k_0}}, f_{k_0}) + \phi(f_{m-k_0}, \overline{f_{m-k_0}}, \nu)} U(f_m \overline{f_{m-k_0}} f_m f_{m-k_0}; f_{k_0} \nu). \end{aligned} \quad (25)$$

Here  $\phi$  is a phase factor and it is a function of the  $U(N)$  irreps. It is shown elsewhere (V.K.B. Kota and Manan Vyas, in preparation) that  $(-1)^{\phi(f_m, \overline{f_{m-k_0}}, f_{k_0}) + \phi(f_{m-k_0}, \overline{f_{m-k_0}}, \nu)}$   $U(- - -)$  will be positive where  $U(- - -)$  is a  $U(N)$   $U$ -coefficient. Therefore we need only  $U^2$  and the formula for this is given by [18],

$$[U(f_m, \overline{f_p}, f_m, f_p; f_{m-p} \nu)]^2 = \frac{\binom{N+1}{\nu}^2 \binom{m-\nu}{p-\nu} \binom{N-\nu-p}{m-p} (N-2\nu+1)}{\binom{N-m+p}{p}^2 \binom{N}{m-p} (N+1)}. \quad (26)$$

Turning to the fourth order moments, we need  $M_{13}$ ,  $M_{31}$  and  $M_{22}$ . As  $\mathcal{O}^\dagger \neq \mathcal{O}$ , here  $M_{13} \neq M_{31}$  [similarly  $M_{40} \neq M_{04}$  and  $M_{20} \neq M_{02}$  as seen from Eq. (22)]. Following the procedure used for deriving the formula for  $M_{11}(m)$ , we have for  $M_{31}(m)$

$$\begin{aligned} M_{31}(m) &= \overline{\langle \mathcal{O}^\dagger H \mathcal{O} H^3 \rangle^m} = 2 \overline{\langle H^2 \rangle^m} M_{11}(m) \\ &+ V_{\mathcal{O}}^2 V_H^2 \binom{N}{m}^{-1} \sum_{v_1, v_2, v_3, v_4, v_5, v_6; \alpha, \nu_1, \omega_{\nu_1}, \nu_2, \omega_{\nu_2}} \\ &\langle m, v_1 | A_\alpha^\dagger(k_0) | m - k_0, v_2 \rangle \langle m - k_0, v_3 | A_\alpha(k_0) | m, v_4 \rangle \\ &\times \langle m - k_0, v_2 | B^{\nu_1, \omega_{\nu_1}}(k) | m - k_0, v_3 \rangle \langle m, v_5 | B^{\nu_1, \omega_{\nu_1}}(k) | m, v_6 \rangle \\ &\times \langle m, v_4 | B^{\nu_2, \omega_{\nu_2}}(k) | m, v_5 \rangle \langle m, v_6 | B^{\nu_2, \omega_{\nu_2}}(k) | m, v_1 \rangle. \end{aligned} \quad (27)$$

Now, applying the Wigner-Eckart theorem, using the results in Section 2 and simplifying the resulting C-G coefficients will give,

$$\begin{aligned} M_{31}(m) &= \overline{\langle \mathcal{O}^\dagger H \mathcal{O} H^3 \rangle^m} = 2 \overline{\langle H^2 \rangle^m} M_{11}(m) \\ &+ V_{\mathcal{O}}^2 V_H^2 \binom{N}{m}^{-1} \binom{N-k_0}{m-k_0} \sum_{\nu=0}^{\min(k, m-k)} \Lambda^\nu(N, m, k) Z_{11}(N, m, k_0, k, \nu). \end{aligned} \quad (28)$$

The function  $Z_{11}$  is defined in Eq. (25). Following the same procedure as above, the formula for  $M_{13}$  is,

$$\begin{aligned} M_{13}(m) &= \overline{\langle \mathcal{O}^\dagger H^3 \mathcal{O} H \rangle^m} = 2 \overline{\langle H^2 \rangle^{m-k_0}} M_{11}(m) \\ &+ V_{\mathcal{O}}^2 V_H^2 \binom{N}{m}^{-1} \binom{N-k_0}{m-k_0} \sum_{\nu=0}^{\min(k, m-k_0-k)} \Lambda^\nu(N, m - k_0, k) Z_{11}(N, m, k_0, k, \nu). \end{aligned} \quad (29)$$

Derivation of the formula for  $M_{22}$  is more involved. Leaving details to a long paper under

preparation, the final result (with  $\rho$  a multiplicity label) is

$$\begin{aligned}
M_{22}(m) &= \overline{\langle \mathcal{O}^\dagger H^2 \mathcal{O} H^2 \rangle^m} = \overline{\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m} \overline{\langle H^2 \rangle^m} \overline{\langle H^2 \rangle^{m-k_0}} \\
&+ V_{\mathcal{O}}^2 V_H^2 \left\{ \binom{N}{m} \binom{N}{k_0} \right\}^{-1} \binom{N-k_0}{m-k_0} \left\{ \sum_{\nu=0}^k Z_{11}(N, m, k_0, k, \nu) \right\}^2 \\
&+ V_{\mathcal{O}}^2 V_H^2 \binom{N}{m}^{-1} \binom{N-k_0}{m-k_0} \sum_{\nu_1=0}^k \sum_{\nu_2=0}^k \sum_{\nu=0}^{2k} \sqrt{\binom{N}{k_0}} d(N : \nu) \\
&\times \sum_{\rho} \langle m \parallel [B^{\nu_1}(k) B^{\nu_2}(k)]^{\nu, \rho} \parallel m \rangle \langle m - k_0 \parallel [B^{\nu_1}(k) B^{\nu_2}(k)]^{\nu, \rho} \parallel m - k_0 \rangle \\
&\times (-1)^{\phi(f_m, \overline{f_{m-k_0}}, k_0) + \phi(f_{m-k_0}, \overline{f_{m-k_0}}, \nu)} U(f_m \overline{f_{m-k_0}} f_m f_{m-k_0}; f_{k_0} \nu) .
\end{aligned} \tag{30}$$

The moments  $M_{PQ}$  can be converted into reduced (scale free) cumulants  $k_{PQ}$  that gives information about the shape of the bivariate transition strength density. For our purpose the first non-trivial cumulants are the fourth order cumulants and they are given by,

$$\begin{aligned}
k_{40} &= \mu_{40} - 3, \quad k_{04} = \mu_{04} - 3, \\
k_{31} &= \mu_{31} - 3\xi, \quad k_{13} = \mu_{13} - 3\xi, \\
k_{22} &= \mu_{22} - 2\xi^2 - 1; \\
\mu_{PQ} &= \left\{ \left[ \widetilde{M}_{20} \right]^{P/2} \left[ \widetilde{M}_{02} \right]^{Q/2} \right\}^{-1} \widetilde{M}_{PQ} \quad \text{and} \quad \widetilde{M}_{PQ} = M_{PQ}/M_{00} .
\end{aligned} \tag{31}$$

The  $k_{PQ}$ ,  $P + Q = 4$  follow from Eqs. (12), (14), (18), (22), (25), (28), (29) and (30). Numerical results for some typical values of  $(N, m, k, k_0)$  are shown in Table 1. These results show that in general  $|k_{PQ}| \lesssim 0.3$  indicating that the bivariate strength density will be close to a bivariate Gaussian. For further confirming this result, we will derive asymptotic results for  $k_{PQ}$ .

#### 4. Asymptotic formulas for bivariate moments and approach to bivariate Gaussian form

Here we will consider the asymptotic limit defined by  $N \rightarrow \infty$  with  $m, k$  and  $k_0$  fixed and  $k, k_0 \ll m$ . Note that in the dilute limit (or true asymptotic limit) we also have  $m \rightarrow \infty$  and  $m/N \rightarrow 0$  with  $k$  and  $k_0$  fixed. Using the formulas given in Sections 2 and 3 first we can show that in the asymptotic limit (asympt),

$$\begin{aligned}
\binom{N}{m}^{-1} \binom{N-k_0}{m-k_0} Z_{11}(N, m, k_0, k, k) &= \left[ \binom{N}{k_0} d(N : k) \Lambda^k(N, m, m - k) \Lambda^k(N, m - k_0, m - k_0 - k) \right]^{1/2} \\
&\times \binom{N}{m}^{-1} \binom{N-k_0}{m-k_0} |U(f_m, \overline{f_{m-k_0}}, f_m, f_{m-k_0}; f_{k_0}, k)| \xrightarrow{\text{asympt}} \binom{m}{k} \binom{N}{k} \binom{m-k}{k_0}, \\
\overline{\langle [H(k)]^2 \rangle^m} &= \Lambda^0(N, m, k) \xrightarrow{\text{asympt}} \binom{m}{k} \binom{N}{k}, \\
\binom{N}{m}^{-1} \Lambda^k(N, m, m - k) \Lambda^k(N, m, k) d(N : k) &\xrightarrow{\text{asympt}} \binom{m}{k} \binom{m-k}{k} \binom{N}{k}^2 .
\end{aligned} \tag{32}$$



**Table 1.** Bivariate correlation coefficient ( $\xi$ ) and fourth order bivariate cumulants  $k_{rs}$  with  $r \geq s$  and  $r + s = 4$  for various values of number of sp states ( $N$ ), number of fermions ( $m$ ), Hamiltonian body rank ( $k$ ) and the rank ( $k_0$ ) of the particle removal transition operator. Results are obtained using the formulas given in Section 3. Note that for the  $M_{22}$  that is needed for  $k_{22}$ , we have used Eq. (30) with the third term replaced by the corresponding asymptotic formula given by Eq. (37) as a formula for the reduced matrix elements in Eq. (30) is not available.

$N$	$m$	$k$	$k_0$	$\xi$	$k_{40}$	$k_{04}$	$k_{31}$	$k_{13}$	$k_{22}$
20	10	2	1	0.82	-0.54	-0.55	-0.44	-0.45	-0.21
30	10	2	1	0.85	-0.48	-0.50	-0.41	-0.43	-0.26
60	10	2	1	0.88	-0.42	-0.46	-0.37	-0.40	-0.30
80	10	2	1	0.88	-0.41	-0.45	-0.36	-0.39	-0.31
50	12	2	1	0.89	-0.38	-0.40	-0.34	-0.36	-0.25
	15	2	1	0.91	-0.33	-0.35	-0.30	-0.31	-0.19
	20	2	1	0.92	-0.29	-0.29	-0.26	-0.27	-0.13
	25	2	1	0.92	-0.27	-0.27	-0.25	-0.25	-0.08
24	8	2	1	0.82	-0.56	-0.61	-0.46	-0.49	-0.31
		2	2	0.66	-0.56	-0.67	-0.37	-0.43	-0.22
40	15	2	1	0.90	-0.36	-0.37	-0.32	-0.33	-0.18
		2	2	0.80	-0.36	-0.38	-0.29	-0.31	-0.12
60	20	2	1	0.93	-0.27	-0.27	-0.25	-0.25	-0.14
		3	1	0.89	-0.51	-0.53	-0.46	-0.47	-0.30
		3	2	0.79	-0.51	-0.54	-0.40	-0.43	-0.22

Starting with  $\xi$ , it should be clear that in the asymptotic limit only the term with  $\nu = k$  in Eq. (25) will survive. Then, applying the first two relations in Eq. (32) will give

$$\xi(m) = \frac{M_{11}(m)}{M_{00}(m) \left[ \widetilde{M}_{20}(m) \widetilde{M}_{02}(m) \right]^{1/2}} \quad (33)$$

$$\xrightarrow{\text{asympt}} \frac{\binom{m-k}{k_0} \binom{m}{k}^{1/2}}{\binom{m}{k_0} \binom{m-k_0}{k}^{1/2}} .$$

Similarly, for  $k_{40}$  and  $k_{04}$  only the terms with  $\nu = k$  in Eq. (14) will survive and then applying the second and third relations in Eq. (32) will give,

$$k_{40}(m) = \frac{\widetilde{M}_{40}(m)}{\left[ \widetilde{M}_{20}(m) \right]^2} - 3 \xrightarrow{\text{asympt}} \frac{\binom{m-k}{k}}{\binom{m}{k}} - 1 , \quad (34)$$

$$k_{04}(m) = \frac{\widetilde{M}_{04}(m)}{\left[ \widetilde{M}_{02}(m) \right]^2} - 3 \xrightarrow{\text{asympt}} \frac{\binom{m-k_0-k}{k}}{\binom{m-k_0}{k}} - 1 .$$

For  $M_{31}$ , the first term in Eq. (28) is trivial and in the sum in the second term only the  $\nu = k$  term will survive in the asymptotic limit. Now, applying Eqs. (32) and (7) will give the result

for  $k_{31}(m)$ ,

$$k_{31}(m) \xrightarrow{\text{asympt}} \frac{\binom{m-k}{k} \binom{m-k}{k_0}}{\binom{m}{k_0} \sqrt{\binom{m}{k} \binom{m-k_0}{k}}} - \xi(m) = \xi(m) k_{40}(m) . \quad (35)$$

Similarly  $k_{13}(m)$  is given by,

$$k_{31}(m) \xrightarrow{\text{asympt}} \frac{\binom{m-k_0-k}{k} \binom{m-k}{k_0} \binom{m}{k}^{1/2}}{\binom{m}{k_0} \binom{m-k_0}{k}^{3/2}} - \xi(m) = \xi(m) k_{04}(m) . \quad (36)$$

Finally, in  $M_{22}$  only the third term in Eq. (30) is complicated. This is simplified using its relation, valid in the asymptotic limit, to  $\xi(m)$  as described in [10]. Following this we have for  $k_{22}$ ,

$$\begin{aligned} k_{22}(m) &\xrightarrow{\text{asympt}} -2 [\xi(m)]^2 + \frac{\binom{m}{k} \binom{m-k}{k_0}^2}{\binom{m-k_0}{k} \binom{m}{k_0}^2} + \frac{\binom{m-2k}{k_0} \binom{m-k}{k}}{\binom{m}{k_0} \binom{m-k_0}{k}} \\ &\approx -2 [\xi(m)]^2 + \frac{\binom{m-2k}{k_0}}{\binom{m}{k_0} \binom{m-k_0}{k}} \left[ \binom{m}{k} + \binom{m-k}{k} \right] . \end{aligned} \quad (37)$$

In the dilute limit with  $m \rightarrow \infty$  and  $m/N \rightarrow 0$  and expanding the binomials in Eqs. (33) to (37), it is seen that to order  $1/m$  the cumulants  $k_{rs}$ ,  $r+s=4$  will be  $-k^2/m$  (independent of  $k_0$ ) and the correlation coefficient  $\xi(m) \rightarrow 1 - (kk_0)/2m$ . Thus, the cumulants will tend to zero giving bivariate Gaussian form. However, as  $\xi \rightarrow 1$  as  $m \rightarrow \infty$ , in practice it is necessary to add the  $k_{rs}$ ,  $r+s=4$  corrections to the bivariate Gaussian. This is same as the result seen for  $t$ -body transition operators before in [10, 19].

## 5. Extensions to particle addition operators and beta decay type operators

Firstly, particle addition operator is  $\mathcal{O}^+ = \sum_{\alpha} V_{\alpha} A_{\alpha}^{\dagger}(k_0)$  and acting on a  $m$ -particle state it will generate  $m+k_0$  particle states. It is easy to see that the formulation in Section 3 will apply directly to  $\mathcal{O}^+$  operator by appropriately changing everywhere  $m-k_0$  by  $m+k_0$  giving formulas for  $\xi$  and  $M_{rs}$ ,  $r+s=4$ . Explicit formulas are not given here due to lack of space.

Now we will turn to beta decay type operator and for this consider a system with  $m_1$  fermions in  $N_1$  sp states and  $m_2$  fermions in  $N_2$  sp states with  $H$  preserving  $(m_1, m_2)$ . Then, the  $H$  operator, assumed to be  $k$ -body, is given by,

$$H(k) = \sum_{i+j=k} \sum_{\alpha, \beta \in i} \sum_{a, b \in j} V_{\alpha a; \beta b}(i, j) A_{\alpha}^{\dagger}(i) A_{\beta}(i) A_a^{\dagger}(j) A_b(j) , \quad (38)$$

$$V_{\alpha a; \beta b}(i, j) = \langle i, \alpha : j, a \mid H \mid i, \beta : j, b \rangle .$$

Here we are using Greek labels  $\alpha, \beta, \dots$  to denote the many particle states generated by fermions occupying the orbit with  $N_1$  sp states and the Roman labels  $a, b, \dots$  for the many particle states generated by the fermions occupying the orbit with  $N_2$  sp states. For each  $(i, j)$  pair with  $i+j=k$ , we have a matrix  $V(i, j)$  in the  $k$ -particle space and we assume that the  $V(i, j)$  matrices are represented by independent GUE's with their matrix elements being zero centered with variance,

$$\overline{V_{\alpha a; \beta b}(i, j) V_{\alpha' a'; \beta' b'}(i', j')} = V_H^2(i, j) \delta_{ii'} \delta_{jj'} \delta_{\alpha \beta'} \delta_{ab'} \delta_{\beta \alpha'} \delta_{ba'} . \quad (39)$$

It is important to note that the embedding algebra for the EGUE generated by the action of the  $H(k)$  operator on  $|m_1, v_\alpha : m_2, v_a\rangle$  states is the direct sum algebra  $U(N_1) \oplus U(N_2)$ . Thus we have EGUE( $k$ )- $[U(N_1) \oplus U(N_2)]$  ensemble. A beta decay type transition operator is given by

$$\mathcal{O} = \sum_{\alpha, a} O_{\alpha a} A_\alpha^\dagger(k_0) A_a(k_0) ; \quad O_{\alpha a} = \langle k_0, \alpha | \mathcal{O} | k_0, a \rangle . \quad (40)$$

Note that for beta decay  $k_0 = 1$  and for double beta decay  $k_0 = 2$  in Eq. (40). To proceed further, we assume a GUE representation for the  $\mathcal{O}$  matrix in the defining space giving  $O_{\alpha, a}^\dagger O_{\beta, b} = V_{\mathcal{O}}^2 \delta_{\alpha\beta} \delta_{ab}$ . Note that in general the  $\mathcal{O}$  matrix is a rectangular matrix. Now, the ensemble averaged bivariate moments of the transition strength density are  $M_{PQ}(m_1, m_2) = \overline{\langle \mathcal{O}^\dagger H^Q \mathcal{O} H^P \rangle^{(m_1, m_2)}}$ . Note that  $\mathcal{O}$  takes  $(m_1, m_2)$  to  $(m_1 + k_0, m_2 - k_0)$ . Formulas for the moments will follow by applying the formulation for particle removal operator (given in Section 3) in  $m_2$  space and for particle addition operator in  $m_1$  space with appropriate summations over different parts of  $H$ . Using this, formulas are derived for  $M_{PQ}$  with  $P + Q = 2$  and 4 and there will be reported elsewhere.

## 6. Conclusions and future outlook

In this paper, we have presented exact (finite  $N$ ) results for the moments of the transition strength densities generated by particle removal operators, using  $U(N)$  Wigner-Racah algebra for EGUE random matrix ensembles for spinless fermion systems. In particular, formulas for the moments up to fourth order are derived in detail for the Hamiltonian a EGUE( $k$ ) and a  $k_0$  number of particles removal transition operator with its structure coefficients in the defining spaces [ $V_{\alpha_0}$  in Eq. (16)] assumed to be independent Gaussian variables. Numerical results on one hand and the asymptotic results derived from the exact results on the other, showed that the fourth order cumulants approach zero in the dilute limit implying that the strength densities approach bivariate Gaussian form. As discussed briefly in Section 5, the formulation given in Sections 3 extends to transition operators that are particle addition operators and also to beta decay and neutrinoless double beta decay type operators. Results of the present work, the results reported in the Ghent meeting in June 2014 [10] where the transition operator is a  $t$ -body operator represented by EGUE( $t$ ) and the results (briefly reported in [11] and to be reported in detail in a long paper in preparation) for beta decay and double beta decay type operators establish clearly that the form for the bivariate transition strength densities for isolated finite fermion systems will be *generically a bivariate Gaussian*. Therefore, the bivariate Gaussian form with some corrections can be used in practical applications in calculating transition strengths in complex systems just as the corresponding results for level densities are being applied for calculating nuclear level densities by the Michigan group [20, 21]. Finally, further extensions of the present work to EGUE with  $U(\Omega) \times SU(r)$  embedding discussed in [22] and also to boson systems will be important and they will have applications in mesoscopic systems [here  $r = 2$  with fermions is important] and Bose gases [here  $r = 1$  and  $r = 3$  with bosons will be important].

## Acknowledgments

Thanks are due to Manan Vyas for many useful discussions.

## References

- [1] Brezin E, Kazakov A, Serban D, Wiegmann P and Zabrodin A (editors) 2006 *Applications of Random Matrices in Physics* (The Netherlands: Springer)
- [2] Polkovnikov A, Sengupta K and Silva A 2011 *Rev. Mod. Phys.* **83** 863
- [3] Kota V K B 2001 *Phys. Rep.* **347** 223
- [4] Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 *Rev. Mod. Phys.* **53** 385

- [5] Benet L and Weidenmüller H A 2003 *J. Phys. A: Math. Gen.* **36** 3569
- [6] Papenbrock T and Weidenmüller H A 2007 *Rev. Mod. Phys.* **79** 997
- [7] Gómez J M G, Kar K, Kota V K B, Molina R A, Relaño A and Retamosa J 2011 *Phys. Rep.* **499** 103
- [8] Kota V K B 2014 *Embedded Random Matrix Ensembles in Quantum Physics* (Heidelberg: Springer)
- [9] Weidenmüller H A 2014 *J. Phys. G: Nucl. Part. Phys.* **41** 094010
- [10] Kota V K B 2014 *preprint* arXiv:1411.6391 [math-ph]
- [11] Vyas Manan and Kota V K B 2011 *Preprint* arXiv:1106.0395 [nucl-th]
- [12] Schiffer J P et al 2012 *Phys. Rev. Lett.* **108** 022501
- [13] Freeman S J 2007 *Phys. Rev. C* **75** 051301(R)
- [14] Kota V K B 2005 *J. Math. Phys.* **46** 033514
- [15] Zelevinsky V 2009 *Phys. Atomic Nuclei* **72** 1107
- [16] Furnstahl R J and Hebeler K 2013 *Rep. Prog. Phys.* **76** 126301
- [17] Petrov D S 2014 *Phys. Rev. Lett.* **112** 103201
- [18] Hecht K T 1975 *Comm. Math. Phys.* **41** 135
- [19] French J B, Kota V K B, Pandey A and Tomsovic S 1988 *Ann. Phys. (N.Y.)* **181** 235
- [20] Sen'kov R A, Horoi M and Zelevinsky V 2013 *Comp. Phys. Comm.* **184** 215
- [21] Sen'kov R A, Zelevinsky V and Horoi M 2014 *AIP Conf. Proc.* **1619** 152
- [22] Vyas Manan and Kota V K B 2012 *J. Math. Phys.* **53** 123303